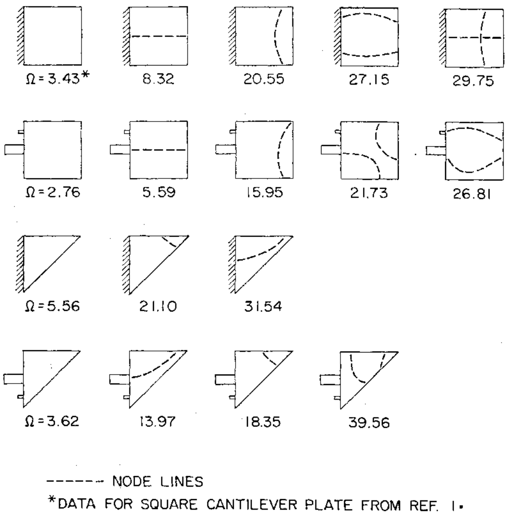

Fig. 1 Frequencies and nodal patterns for vibrating plates.

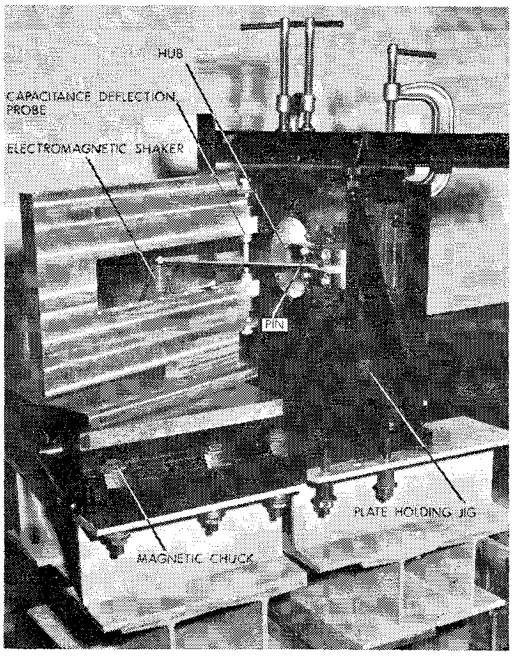

Fig. 2 Hub-pin plate mounting jig and capacitance deflection probe.

portion of the bearing-mounted hub may also be seen in Fig. 2. The deflection probe was supported by a magnetic chuck, which served to provide a rigid base and a convenient means of positioning the probe at various locations on the test plate. The entire deflection probe support was isolated from the base that supported the test plate and the electromagnetic shaker. The electromagnetic shaker and deflection probe are described more fully in Refs. 3 and 4.

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Weak Radiation from a Hypersonic Shock

INGE L. RYHMING*
Aerospace Corporation, El Segundo, Calif.

A REGULAR shock in a one-dimensional supersonic flow may be regarded as a discontinuity in the temperature T, the velocity u, etc., such that the shock can be described in terms of step functions. However, a step-function behavior of the temperature of the medium produces a radiative heat flux. Assuming, for simplicity, that the medium is grey, the radiative flux can be calculated from the formula[1]

$$\bar{q}(\xi) = 2\sigma \int_{-\infty}^{+\infty} T^4 \operatorname{sgn}(\xi' - \xi) E_2(|\xi - \xi'|) d\xi' \quad (1)$$

which defines $\bar{q}(\xi)$ as a radiative flux in the negative x direction; σ is the Stefan-Boltzmann constant, ξ is the optical depth, and $E_2(x)$ is a particular form of the integro-exponential function $E_n(x)$ of general order n (see Ref. 1, p. 253). If the jump of a temperature step function is located at $\xi = 0$, one readily obtains from Eq. (1) the corresponding flux as given by the formula

$$\bar{q}(\xi) = 2\sigma[T^4]_0 E_3(|\xi|) \quad (2)$$

showing that $\bar{q}(\xi)$ is continuous and symmetric with respect to $\xi = 0$. In Eq. (2) the symbol $[\ \ ]_0$ stands for the jump in the quantity within the brackets across the discontinuity, i.e.,

$$[T^4]_0 = T_{0+}{}^4 - T_{0-}{}^4 \quad (3)$$

For a shock wave in cool air, the ratio of the maximum value of $\bar{q}$, which according to Eq. (2) occurs for $\xi = 0$, and the kinetic energy of the freestream, i.e., $\sigma[T^4]_0/\frac{1}{2}\rho_\infty U_\infty{}^3$, is small for a wide range of flow conditions of practical interest. We shall limit the discussion here to such cases. It is then clear that the weak radiative flux produced by the discontinuity in the temperature across the shock will cause the flow ahead of and behind the shock to change, such that profiles are obtained in all flow quantities. We shall calculate these profiles using a simple perturbation analysis of a hypersonic shock in a cool gas.

Taking the governing gasdynamic equations in the hypersonic approximation, and assuming for simplicity a perfect gas with constant specific heats, one can readily derive the following integral equation for the velocity across the shock

$$\frac{2\gamma}{\gamma - 1} u - \frac{\gamma + 1}{\gamma - 1} u^2 = 1 + \frac{4\sigma U_\infty{}^5}{\rho_\infty R^4} \int_{-\infty}^{+\infty} \times (u - u^2)^4 \operatorname{sgn}(\xi' - \xi) E_2(|\xi - \xi'|) d\xi' \quad (4)$$

where γ is the ratio of the specific heats, R the gas constant, U_∞ the unperturbed freestream velocity, ρ_∞ the unperturbed freestream density, and u is a nondimensional velocity equal to the actual velocity divided by U_∞. An equation similar in form to Eq. (4), but applicable in the entire supersonic range was derived and fully investigated by Heaslet and Baldwin.[2] If the radiative flux term is disregarded in Eq. (4), the velocity ahead of and behind the shock have the well-known values

$$\xi < 0: \ u = 1 \qquad \xi > 0: \ u = (\gamma - 1)/(\gamma + 1)$$

A weak radiative flux will perturb these values, such that we

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* Member of the Technical Staff, Aerophysics Department, Aerodynamics and Propulsion Research Laboratory. Member AIAA.

obtain the profiles

$$\text{for } \xi < 0: \ u = 1 - f(\xi) \qquad f(\xi) \ll 1 \qquad (5a)$$

$$\text{for } \xi > 0: \ u = [(\gamma - 1)/(\gamma + 1)][1 + g(\xi)] \qquad g(\xi) \ll 1 \qquad (5b)$$

Relations between the functions f and g are obtained by studying Eq. (4) in the two ranges of ξ specified for the functions f and g, i.e., for $\xi < 0$ and $\xi > 0$, respectively. Thus, for $\xi < 0$, the expression for u according to Eq. (5a) is inserted on the left-hand side of Eq. (4), whereas in the integral term the appropriate expression for u has to be used in the proper range of the integration variable. By treating the case $\xi > 0$ in a similar fashion and by keeping terms to first order in f and g only in both equations, one obtains the following two equations for f and g:

$$\xi < 0: \ f(\xi) = \frac{\gamma - 1}{\gamma + 1}\left[KE_3(-\xi) + \lambda \int_0^\infty g(\xi')E_2(\xi' - \xi)d\xi'\right] \qquad (6a)$$

$$\xi > 0: \ g(\xi) = KE_3(\xi) + \lambda \int_0^\infty g(\xi') \operatorname{sgn}(\xi' - \xi)E_2(|\xi - \xi'|)d\xi' \qquad (6b)$$

where

$$K = [32\sigma U_\infty^5(\gamma - 1)^4]/[\rho_\infty R^4(\gamma + 1)^7] \qquad \lambda = 2(3 - \gamma)K \qquad (7)$$

Equation (6b) is a linear Fredholm equation of the second kind for the function g, and the function f is related to the function g through the simple quadrature formula given by Eq. (6a).

In studying Eq. (6b), we notice first that it is singular in the sense that the norm of the kernel is unbounded, i.e.,

$$\int_0^\infty \int_0^\infty E_2^2(|\xi - \xi'|)d\xi' \, d\xi = \infty$$

Consequently, it is impossible to use the standard theory[3] to obtain the solution, and, in particular, to determine the radius of convergence of λ in the Neumann series of the equation. Nevertheless, we shall find it useful, for other purposes to be shown subsequently, to determine the beginning of the Neumann series, which becomes

$$g(\xi) = K\{E_3(\xi) + \lambda[-G_{32}'(\xi) + G_{32}''(\xi)] + 0(\lambda^2)\} \qquad (8)$$

The $G_{mn}'(\xi)$ and $G_{mn}''(\xi)$ are certain functions, which are related to the $E_n(\xi)$ functions, and which also occur in the astrophysical literature (see Ref. 1, p. 259). We notice that at $\xi = 0$, g takes on the value

$$g(0) = K[\tfrac{1}{2} + (\lambda/8) + 0(\lambda^2)] \qquad (9)$$

Because of the singular behavior of Eq. (6b), we shall solve this equation by an approximate method. To this end the $E_n(\xi)$ functions are first simplified in the equation by letting $E_2(x) = m\exp(-nx)$ and $E_3(x) = mn^{-1}\exp(-nx)$ (m and n constants). Next, by taking two derivatives of the simplified equation, and by using the simplified equation itself to eliminate the integrals in the resulting twice differentiated equation, the following differential equation for g, equivalent to the simplified integral equation for g, is obtained:

$$g'' + 2m\lambda g' - n^2 g = 0 \qquad (10)$$

The solution is of the form

$$g = Ae^{-\mu n\xi} \qquad \mu = (m/n)\lambda + \{1 + [(m/n)\lambda]^2\}^{1/2} \qquad (11)$$

where A is a constant, which is formed by inserting the solution for g into the simplified integral equation. One obtains

$$A = (\mu - 1)/2(3 - \gamma)$$

To the same order of approximation the function f can now be evaluated, and the profiles f and g thus become

$$(\xi < 0): = [(\gamma - 1)/(\gamma + 1)]K(m/n)[2\mu/(1 + \mu)]e^{n\xi} \qquad (12a)$$

$$(\xi > 0): \ g = [(\mu - 1)/2(3 - \gamma)]e^{-\mu n\xi} \qquad (12b)$$

To the same order of approximation, the radiative heat flux q, represented in its exact nondimensional form by the integral term in Eq. (4), is found to have the profile

$$\xi < 0: \ q = [2/(\gamma + 1)]K(m/n)[2\mu/(1 + \mu)]e^{n\xi} \qquad (13a)$$

$$\xi > 0: \ q = [2/(\gamma + 1)]K(m/n)[2\mu/(1 + \mu)]e^{-\mu n\xi} \qquad (13b)$$

The Eqs. (13a) and (13b) show that q is continuous at $\xi = 0$, as indeed must be the case.

A solution to the problem is thus obtained, and it remains to assess its applicability. However, to do so, it is first necessary to identify the undetermined constants m and n in the approximation of the $E_n(\xi)$ functions. Evidently, there is no unique way of determining these constants, but certain properties of the exact solution around $\xi = 0$ can be used to indicate which values of m and n would closely approximate the exact solution of Eq. (6b), at least in the neighborhood of $\xi = 0$. To this end, we observe first that the derivative of the flux q with respect to ξ is discontinuous at $\xi = 0$. From the conservation equations it is possible to derive an exact relation for this jump in $dq/d\xi$ at $\xi = 0$.[4] In the present case one obtains

$$[dq/d\xi]_0 = -(8\sigma U_\infty^5/\rho_\infty R^4)[(u - u^2)^4]_0 \qquad (14)$$

By using Eqs. (5) and (12) to calculate the value of the bracketed quantity in Eq. (14) to first order in g and f, one gets

$$[dq/d\xi]_0 = -4\mu K/(\gamma + 1) \qquad (15)$$

The jump in $dq/d\xi$ can also be derived directly from Eq. (13). The value obtained in this way differs from that given by Eq. (15) by the factor m, suggesting the value $m = 1$ for m. Furthermore, for $\xi = 0$, the Neumann series of Eq. (6b) has the form given by Eq. (9). By expanding the approximate solution given by Eq. (12b) for small values of λ and at $\xi = 0$, one obtains an expansion that, with the help of Eq. (7), can be written as

$$g(0) = K[(m/n) + \tfrac{1}{2}(m/n)^2\lambda + 0(\lambda^3)] \qquad (16)$$

Upon comparing Eqs. (16) and (9) we find agreement to $0(\lambda^2)$ within the brackets if $m/n = \frac{1}{2}$, suggesting the value $n = 2$ for n. With these values of m and n, it can be expected that the approximate solution given by Eqs. (12) and (13) gives a good representation of the exact solution to Eq. (6b) close to the shock itself.

The range of applicability of the solution can now be easily determined. In order for the perturbation scheme to be meaningful, it is necessary that $g_{\max} \ll 1$. By assuming that we are dealing with air ($\gamma = 1.4$), it then follows from Eqs. (12b, 11, and 7) that the theory is applicable provided

$$\frac{U_\infty^5}{\rho_\infty} \leq \frac{R^4(\gamma + 1)^7}{16\sigma(\gamma - 1)^4} g_{\max} = 1.36 \times 10^{33} g_{\max} \ (cm^8 s^{-5} g^{-1})$$

Allowing $g_{\max} = 0.1$ in order to have second- and higher-order terms in g be negligible in the theory, and by using a standard density $\rho_\infty = 10^{-3} gcm^{-3}$, the maximum allowable speed from this formula is $U_\infty \simeq 6.7 \times 10^3 \ ms^{-1}$. At a lower density, say $\rho_\infty = 10^{-5} gcm^{-3}$, the corresponding speed is $U_\infty \simeq 2.7 \times 10^3 \ ms^{-1}$. Real gas properties can easily be accounted for in the present context by allowing γ to have an effective value $\gamma_e (\gamma_e < \gamma)$ behind the shock. The result of the analysis is then that γ should be replaced by γ_e in all formulas from Eq. (5) and on, except that, in the formulas for f, Eq. (6a) and (12a), the factor $(\gamma - 1)$ in the numerator remains unchanged. The region of applicability of the theory is then extended further. However, at low densities and high speeds additional complications of nonequilibrium phenomenon occurs, and the results derived previously can then be altered to a substantial degree (see Ref. 4).

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Magnetohydrodynamic Effect on Open-Channel Flow

WALTER J. BORNHORST*
Massachusetts Institute of Technology, Cambridge, Mass.

Introduction

The behavior of a conducting fluid, flowing in the absence of an electric field in an open channel under the influence of gravity and an applied vertical magnetic field, was investigated theoretically and experimentally.

As shown in Fig. 1, the mercury enters the channel with an x-directed uniform velocity (V_0). The effect of the magnetic field is to induce an electric field and a current (j) in the negative z direction. The current returns outside the channel through an artificial perfect conductor. Since the current is transverse to the magnetic field (B), a $j \times B$ body force is induced in the negative x direction. The approximate values of the independent parameters for the cases tested are as follows: velocity (V_0) = 1 cm/sec, channel height (Y_0) = 7.5 cm, channel length (L_0) = 15 cm, and magnetic field (B) = 10 kgauss.

These values of the independent parameters are used with mercury as the conducting fluid. The following assumptions are reasonable: first, all of the induced magnetic fields are neglected (the magnetic Reynolds number is of the order of 10^{-3}); second, the flow is assumed inviscid (the Reynolds numbers under consideration are of the order of 10^4); third, since the displacement thickness is small as compared to the characteristic channel dimensions, the influence of the boundary layer is neglected; fourth, inertia terms are neglected, in that $V_0^2/y_0 g$ is of the order of 10^{-4}.

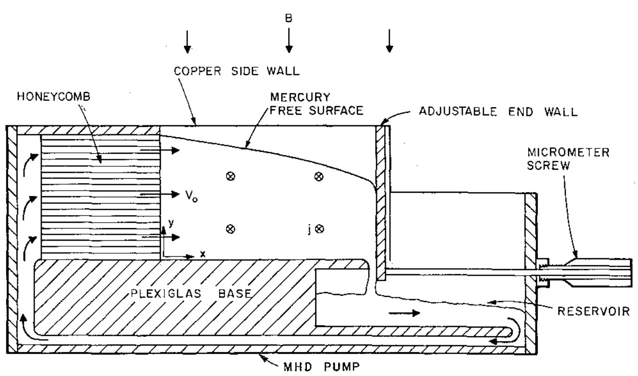

Fig. 1 Sectional schematic of flow system.

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* Research Assistant, Department of Mechanical Engineering.

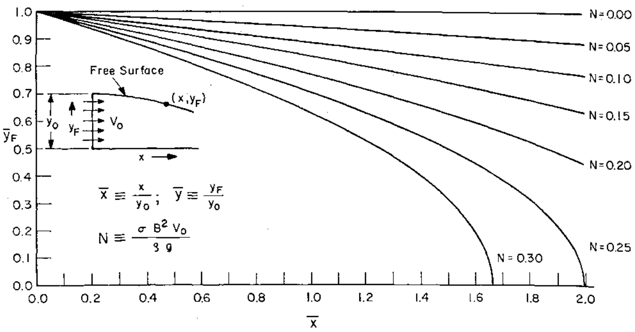

Fig. 2 Theoretical curves of free surface.

Theoretical Analysis

The linearized momentum equation for the steady flow of an inviscid and incompressible fluid is, in this case,

$$\nabla P + \sigma u B^2 i_x + \rho g i_y = 0 \qquad (1)$$

where P is the pressure, σ the fluid conductivity, u the x component of velocity, ρ the mass density, g the acceleration of gravity, and i_x and i_y are unit vectors in the x and y directions, respectively.

By taking the curl of Eq. (1), it readily is seen that $\partial u/\partial y$ must vanish. Hence, the continuity equation can be written as

$$u y_F = V_0 y_0 \qquad (2)$$

where y_F is the vertical distance from the bottom of the channel to the free surface. The momentum equation written along a streamline is

$$(1/\rho)dP + (\sigma u B^2/\rho)(dx)_s + g(dy)_s = 0 \qquad (3)$$

where $(dx)_s$ and $(dy)_s$ are such that $(dy)_s/(dx)_s$ is the slope of the streamline. Writing Eq. (3) for the free surface streamline and substituting for u from Eq. (2) yields

$$(dy)_F/(dx)_F = -(\alpha B^2 V_0 y_0/\rho y_F) \qquad (4)$$

where $(dy)_F/(dx)_F$ is the slope of the free surface streamline. Integration of Eq. (4) from 0, y_0 to some arbitrary point x, y_F, and subsequent normalization yields the equation for the free surface. Thus

$$\bar{y}_F = (1 - 2N\bar{x})^{1/2} \qquad (5)$$

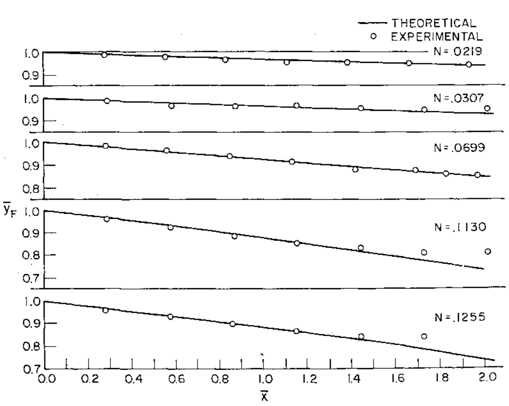

Fig. 3 Comparison of experimental and theoretical free surfaces.